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RAPID COMPUTATION OF FLEXURAL CONSTANTS

By Thomas G. Morrison, A. M. ASCE

STRUCTURAL DIVISION

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AMERICAN SOCIETY OF CIVIL ENGINEERS

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PAPERS

RAPID COMPUTATION OF FLEXURAL CONSTANTS

By Thomas G. Morrison,[1] A. M. ASCE

Synopsis

Short algebraic formulas for computing stiffness, fixed-end moments, and carry-over factors in terms of three functions and their components are given. This paper has attempted to systematize and thereby simplify the evaluation of these factors, by the introduction of the flexural functions. A numerical method for rapidly computing the values of the functions for nonprismatic beams is presented.

Introduction

Computation of the flexural constants for nonprismatic beams by the usual methods involves considerable hidden repetition in the numerical work. This repetition indicates the existence of simpler functions, of which the flexural constants are composed. Investigation discloses that the flexural constants can be computed by combining a system of three functions that cannot be simplified further.

In this paper, all quantities are taken as positive and evident physical relationships are relied upon to keep the signs of moments correct.

Definitions

Notation.—The letter symbols adopted for use in this paper are defined where they first appear, in the illustrations or in the text, and are arranged alphabetically for reference in the Appendix.

The three flexural functions are defined as:

$$\left.\begin{aligned}
\Phi_0 &= \int_0^1 \frac{I_a}{I_x}\, dx \\[4pt]
\Phi_1 &= \int_0^1 \frac{I_a}{I_x}\, x\, dx \\[4pt]
\Phi_2 &= \int_0^1 \frac{I_a}{I_x}\, x^2\, dx
\end{aligned}\right\} \quad \dots\dots\dots\dots\dots\dots(1)$$

NOTE.—Written comments are invited for publication; the last discussion should be submitted by **August 1, 1953.**

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in which I_a is the moment of inertia evaluated at the small end of the beam, and I_z is the moment of inertia at any section. The distance $x L$ is indicated in Fig. 1.

For computing the fixed-end moment coefficients, additional components of Φ_0, Φ_1, and Φ_2 will be required, corresponding to each load position $r L$:
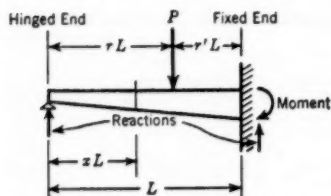

FIG. 1.—BEAM OF VARYING CROSS SECTION

$$\left.\begin{aligned}\Phi_{0,r} &= \int_r^1 \frac{I_a}{I_z}\, dx \\[6pt] \Phi_{1,r} &= \int_r^1 \frac{I_a}{I_z}\, x\, dx \\[6pt] \Phi_{2,r} &= \int_r^1 \frac{I_a}{I_z}\, x^2\, dx\end{aligned}\right\} \quad\dots\dots\dots\dots (2)$$

If the computations are carried out by numerical integration, no significant additional work is required to obtain the functions of Eqs. 2.

For use in computing $\Phi_{0,r}$, $\Phi_{1,r}$, and $\Phi_{2,r}$ by numerical integration, the following factors are defined:

$$\left.\begin{aligned}\Phi'_{0,r} &= \int_r^{r+\frac{1}{n}} dx \\[6pt] \Phi'_{1,r} &= \int_r^{r+\frac{1}{n}} x\, dx \\[6pt] \Phi'_{2,r} &= \int_r^{r+\frac{1}{n}} x^2\, dx\end{aligned}\right\} \quad\dots\dots\dots\dots (3)$$

In Eqs. 3, n is the number of intervals into which the beam is divided for numerical integration. By Eq. 3, $\Phi'_{0,r}$ is the reciprocal of n.

A fourth function can be defined as

$$\Phi_3 = \int_0^1 \frac{I_a}{I_z} x^3\, dx \dots\dots\dots\dots\dots (4a)$$

The function in Eq. 4a is convenient for computing the fixed-end moment factors for a beam carrying a uniformly distributed load. The factors $\Phi'_{3,r}$ are defined as follows:

$$\Phi'_{3,r} = \int_r^{r+\frac{1}{n}} x^3\, dx \dots\dots\dots\dots\dots (4b)$$

These factors can be tabulated for various n-values and are convenient for computing Φ_3 by numerical integration.

An infinite set of functions—

$$\Phi_m = \int_0^1 \frac{I_a}{I_z} x^m\, dx \dots\dots\dots\dots\dots (5)$$

—can be defined. In Eq. 5, m is any integer. The functions up to and including Φ_m could be used to compute the fixed-end moment factors for a distributed load varying as x^{m-2}. Actually, for such loading it is easier to sum the fixed-end moment factors for concentrated loading, each weighted by the proper factor.

WORKING FORMULAS

Having computed the flexural functions, the coefficients for the flexural constants can be computed from the following formulas, which can be easily derived by the moment-area method or by a similar method. The carry-over factor at the small end is

$$C_{ab} = \frac{(\Phi_1 - \Phi_2)}{\Phi_2}$$

—and at the large end

$$C_{ba} = \frac{(\Phi_1 - \Phi_2)}{(\Phi_0 - \Phi_1) - (\Phi_1 - \Phi_2)} \Bigg\} \dots\dots\dots\dots\dots (6)$$

The stiffness factors are

$$\begin{aligned} k_a &= (\Phi_0 - \Phi_1) - C_{ab}\,\Phi_1 \\ k_b &= \Phi_1 - C_{ba}\,(\Phi_0 - \Phi_1) \end{aligned} \Bigg\} \dots\dots\dots\dots\dots (7)$$

If N is a quantity defined as follows—

$$N = \frac{\Phi_{2,r} - r\,\Phi_{1,r}}{\Phi_2} \dots\dots\dots\dots\dots\dots (8)$$

then the angle-of-rotation factor at the small end is

$$\theta_{a,r} = N\,\Phi_1 - \Phi_{1,r} + r\,\Phi_{0,r} \dots\dots\dots\dots\dots (9a)$$

The fixed-end moment factor for the beam of Fig. 1 is

$$F'_{b,r} = r' - N \dots\dots\dots\dots\dots\dots (9b)$$

and for a beam having both ends fixed, the fixed-end moment factors are

$$\begin{aligned} F_{a,r} &= \frac{1}{k_a}\,\theta_{a,r} \\ F_{b,r} &= F'_{b,r} - C_{ab}F_{a,r} \end{aligned} \Bigg\} \dots\dots\dots\dots\dots (9c)$$

For uniform loading,

$$\theta_{a,u} = \tfrac{1}{2}\left[\frac{\Phi_3\,\Phi_1}{\Phi_2} - \Phi_2 \right] \dots\dots\dots\dots\dots (10a)$$

$$F'_{b,u} = \tfrac{1}{2}\left[1 - \frac{\Phi_3}{\Phi_2} \right] \dots\dots\dots\dots\dots (10b)$$

$$\begin{aligned} F_{a,u} &= \frac{1}{k_a}\,\theta_{a,u} \\ F_{b,u} &= F'_{b,u} - C_{ab}\frac{1}{k_a}\,\theta_{a,u} \end{aligned} \Bigg\} \dots\dots\dots\dots\dots (10c)$$

If the fixed-end moments for concentrated loads are required, the function Φ_3 need not be computed. Instead, $F_{a,u}$ and $F_{b,u}$ can be calculated from the expressions:

$$\left.\begin{array}{l} F_{a,u} = \dfrac{1}{n-1} \displaystyle\sum_r F_{a,r} \\[3mm] F_{b,u} = \dfrac{1}{n-1} \displaystyle\sum_r F_{b,r} \end{array}\right\} \dots\dots\dots\dots\dots\dots (11)$$

in which n is the number of intervals into which the member is divided for the computation of $F_{a,r}$ and $F_{b,r}$.

FIXED–END MOMENT FACTORS

Usually, $F_{a,r}$ and $F_{b,r}$ are computed directly, and are used for computing $F'_{a,r}$ and $F'_{b,r}$ by algebraic reduction. This procedure is cumbersome, time-consuming, and inaccurate because, to obtain $F_{a,r}$ and $F_{b,r}$ directly, simultaneous equations usually must be solved. The evident reason for the relative difficulty of direct computation of $F_{a,r}$ and $F_{b,r}$ is that of all the flexural constants they alone are defined for doubly indeterminate beams. However, $F'_{a,r}$ and $F'_{b,r}$ are defined for singly indeterminate beams. When flexural functions are employed, the numerical work is the least if $F'_{b,r}$ and $\theta_{a,r}$ are computed directly, and $F_{a,r}$ and $F_{b,r}$ then obtained from these. No steps are wasted by using this procedure.

NUMERICAL INTEGRATION

Although the flexural functions theoretically can be computed by integration, it is well known to engineers that, even for so simple a beam as a slab having a parabolic soffit, evaluation of the resulting integrals becomes quite formidable. Furthermore, for a beam whose soffit is a compound curve, the work is greatly increased. Because of these practical problems, it usually will be found simpler to evaluate the flexural functions by numerical integration. This method has the additional advantage that the components of the functions required for obtaining the fixed-end moment factors can be obtained without significant additional work.

FIG. 2.—STEPPED EQUIVALENT BEAM FOR NUMERICAL INTEGRATION (BEAMS ARE APPROXIMATELY EQUAL)

For purposes of numerical integration, the actual beam is replaced by an approximately equivalent beam consisting of n segments (taken of equal length for convenience only) each segment of constant moment of inertia. This condition is illustrated in Fig. 2. Then,

$$\int_0^1 \frac{I_a}{I_x} x^m \, dx = \sum_{q=r_1}^{q=r_n} \frac{I_a}{I_q} \int_q^{q+\frac{1}{n}} x^m \, dx \dots\dots\dots\dots (12a)$$

or, in terms of flexural functions,

$$\Phi_m = \sum_{q=r_1}^{q=r_n} \frac{I_a}{I_q} \Phi'_{m,\,q} \dots\dots\dots\dots\dots(12b)$$

Similarly,

$$\Phi_{m,\,r} = \sum_{q=r}^{q=r_n} \frac{I_a}{I_q} \Phi'_{m,\,q} \dots\dots\dots\dots\dots(12c)$$

In Eqs. 12, q is the segment designation. Because the factors $\Phi'_{m,\,r}$ are independent of the beam shape, they can be tabulated for various values of n. Tables of $\Phi'_{0,\,r}$, $\Phi'_{1,\,r}$, and $\Phi'_{2,\,r}$ are given for $n = 4$, 6, 8, 10, and 12.

Errors in Numerical Integration.—It should be observed that Eqs. 12 are theoretically accurate. The only approximation in the process is the substitution of the stepped beam for the continuously varying beam as illustrated in Fig. 2. The margin of error is indicated in Table 1, in which the values of

TABLE 1.—Comparison Between Values of Φ Computed by Exact and Approximate Methods

Flexural functions for beam	Theoretical value (exact)	n = 4		n = 8		n = 12	
		Value by numerical method	% error	Value by numerical method	% error	Value by numerical method	% error
Φ_0	0.37500	0.3675	2.00	0.37300	0.53	0.374	0.27
Φ_1	0.12500	0.1260	0.80	0.12564	0.51	0.12536	0.29
Φ_2	0.06815	0.06905	1.32	0.06837	0.32	0.06838	0.34

Φ_0, Φ_1, and Φ_2 have been computed for the beam of continuously varying depth shown and for three stepped, approximately equivalent beams. For the worst case ($n = 4$), Φ_0 is only 2% in error and for $n = 12$, the error is reduced to 0.25%. These values are approximate because they were all computed with a slide rule, but they are therefore indicative of the accuracy to be expected in an actual problem. The percentage of error in the final stiffness, carry-over factor, and fixed-end moment would be three or four times that of the individual functions because of the magnification of the errors by subtraction.

For concrete beams, accuracy greater than 5% is of academic importance only. For steel beams having uniform depth and stepped cover plates, the numerical method involves no intrinsic inaccuracy.

EXAMPLES

Two examples are presented to illustrate the method. For the first, a simple problem is selected that occurs frequently in practice. This is the problem of a prismatic beam having infinitely rigid ends.

The flexural constants for the top beam of the nonsymmetrical bent shown in Fig. 3 are required. For this problem, symbolic integration is easier than numerical integration, and therefore the problem offers a good illustration of the basic theory without the secondary complications of numerical integration.
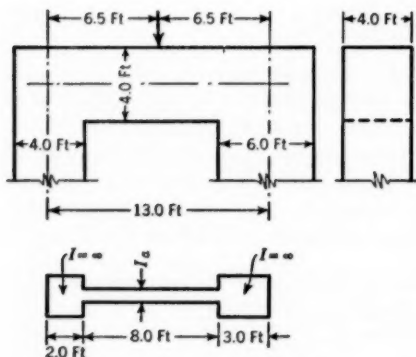

FIG. 3.—NONSYMMETRICAL BENT

As is conventional with most moment-distribution formulas for beams of variable section, the constant value of moment of inertia, "I_a," given in the formulas is that at the small or left-hand end. It should be observed that this choice is quite arbitrary and conventional and that any particular value of I_a might just as well have been selected with the provision only that the same value be used throughout any given set of computations. In the present problem, because the moments of inertia at the ends of the beam are infinite and therefore indeterminate, I_a is taken as the moment of inertia of the central portion.

The limits of integration are $2/13 = 0.154$ and $10/13 = 0.769$. Thus, the flexural functions are as follows:

$$\Phi_0 = \int_0^1 \frac{I_a}{I_x} dx = \int_0^{0.154} \frac{I_a}{\infty} dx + \int_{0.154}^{0.769} \frac{I_a}{I_a} dx + \int_{0.769}^1 \frac{I_a}{\infty} dx$$

$$= 0 + 0.769 - 0.154 + 0 = 0.615 \ldots (13a)$$

Similary,

$$\Phi_1 = \int_{0.154}^{0.769} \frac{I_a}{I_a} x \, dx = 0.284 \ldots \ldots \ldots \ldots (13b)$$

and

$$\Phi_2 = \frac{x^3}{3} \Big|_{0.154}^{0.769} = 0.150 \ldots \ldots \ldots \ldots (13c)$$

For the fixed-end moments produced by a concentrated load at the center of the beam,

$$\Phi_{0,\,0.5} = \int_{0.5}^{0.769} \frac{I_a}{I_a} dx = 0.269$$

Similarly,

$$\Phi_{1,\,0.5} = 0.171$$

and

$$\Phi_{2,\,0.5} = 0.110$$
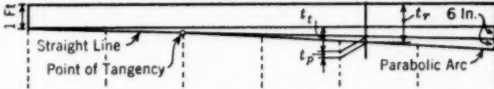
$$\ldots \ldots \ldots \ldots (14)$$

Then, from Eqs. 6,

$$C_{ab} = \frac{\Phi_1 - \Phi_2}{\Phi_2} = \frac{0.284 - 0.150}{0.150} = 0.894$$

$$C_{ba} = \frac{\Phi_1 - \Phi_2}{(\Phi_0 - \Phi_1) - (\Phi_1 - \Phi_2)} = 0.680,$$

and, from Eqs. 7, $k_a = 0.077$ and $k_b = 0.059$. As a check, the known prop-

TABLE 2.—COMPUTATION OF DIMENSIONLESS FACTORS FOR STIFFNESS, FIXED-END MOMENTS AND CARRY-OVER FACTORS FOR A NONPRISMATIC BEAM

Line No.	Term or factor						
1	l_t	0.50	1.50	2.50	3.50	4.50	5.50
2	t_p	0.00	0.00	0.09	0.84	2.34	4.60
3	t_r	12.50	13.50	14.59	16.34	18.84	22.10
4	$I_a/I_r = (t_a/t_r)^3$	1.000	0.795	0.630	0.448	0.292	0.181
5	$\Phi'_{0,r} I_a/I_r$	0.1667	0.1323	0.1050	0.0746	0.0487	0.0302
6	$\Phi_{0,r}$		0.3908	0.2585	0.1535	0.0789	0.0302
7	Φ_0	0.5575					
8	$\Phi'_{1,r} I_a/I_r$	0.0139	0.0331	0.0437	0.0435	0.0365	0.0276
9	$\Phi_{1,r}$		0.1844	0.1513	0.1076	0.0641	0.0276
10	Φ_1	0.1983					
11	$\Phi'_{2,r} I_a/I_r$	0.0015	0.0086	0.0185	0.0256	0.0275	0.0254
12	$\Phi_{2,r}$		0.1056	0.0970	0.0785	0.0529	0.0254
13	Φ_2	0.1071					
14	$\Phi_0 - \Phi_1$	0.3592					
15	$\Phi_1 - \Phi_2$	0.0912					
16	$(\Phi_0 - \Phi_1) - (\Phi_1 - \Phi_2)$	0.2680					
17	By Eqs. 6 and 7: $C_{ab} = 0.851$, $C_{ba} = 0.341$, $k_a = 0.190$, $k_b = 0.076$. Check: $\frac{C_{ab} k_b}{C_{ba} k_a} = 0.999$						
18	$r \Phi_{1,r}$		0.0307	0.0504	0.0538	0.0427	0.0230
19	$\Phi_{2,r} - r \Phi_{1,r}$		0.0749	0.0466	0.0247	0.0102	0.0024
20	N		0.7000	0.435	0.231	0.094	0.022
21	$N \Phi_1$		0.1388	0.0863	0.0458	0.0186	0.0044
22	$r \Phi_{0,r}$		0.0651	0.0862	0.0768	0.0526	0.0252
23	$N \Phi_1 + r \Phi_{0,r}$		0.2039	0.1725	0.1226	0.0712	0.0296
24	$\Theta_{a,r}$		0.0195	0.0212	0.0150	0.0071	0.0020
25	r'		0.8333	0.6667	0.5000	0.3333	0.1667
26	$F'_{b,r}$		0.1333	0.2317	0.269	0.2393	0.1447
27	$F_{a,r}$		0.1028	0.1118	0.0790	0.0347	0.0105
28	$C_{ab} F_{a,r}$		0.0174	0.0189	0.0133	0.0063	0.0018
29	$F_{b,r}$		0.1159	0.2128	0.2557	0.2330	0.1429
30	$\Sigma F_{a,r}$	0.3415					
31	$F_{a,u}$	0.0683					
32	$\Sigma F_{b,r}$	0.9603					
33	$F_{b,u}$	0.1921					

erty $\frac{C_{ab} k_b}{C_{ba} k_a} = 1$ is used. Thus, $\frac{0.894 \times 0.059}{0.680 \times 0.077} = 0.992$, indicating an error of less than 1%.

The actual stiffness factors are (using a value of 1 for E):

$$K_a = \frac{1 \times 4 \times 4.5^3}{12 \times 0.077 \times 13} = 30.4 \text{ kip-ft}$$
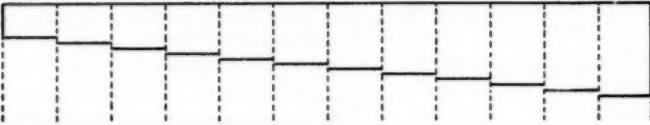
$$K_b = \frac{1 \times 4 \times 4.5^3}{12 \times 0.059 \times 13} = 39.6 \text{ kip-ft}.$$

For the fixed-end moment for a concentrated load at the center of the beam,

$$N = \frac{\Phi_{2,0.5} - r\,\Phi_{1,0.5}}{\Phi_2} = \frac{0.110 - (0.5 \times 0.171)}{0.150} = 0.167$$

and

$$\Phi_{0,0.5} = N\,\Phi_1 - \Phi_{1,0.5} + r\,\Phi_{0,0.5} = 0.167 \times 0.284$$
$$- 0.171 + (0.5 \times 0.269) = 0.0110.$$

n = 12	$\Phi'_{0,r}$	0833											
	$\Phi'_{1,r}$	0035	0104	0174	0243	0313	0382	0451	0521	0590	0660	0729	0799
	$\Phi'_{2,r}$	0002	0014	0037	0071	0118	0176	0245	0326	0418	0523	0639	0766
n = 10	$\Phi'_{0,r}$	1000											
	$\Phi'_{1,r}$	0050	0150	0250	0350	0450	0550	0650	0750	0850	0950		
	$\Phi'_{2,r}$	0003	0023	0063	0123	0203	0303	0423	0563	0723	0903		
n = 8	$\Phi'_{0,r}$	1250											
	$\Phi'_{1,r}$	0078	0234	0391	0547	0703	0859	1016	1172				
	$\Phi'_{2,r}$	0007	0046	0124	0241	0397	0592	0827	1100				
n = 6	$\Phi'_{0,r}$	1667											
	$\Phi'_{1,r}$	0139	0417	0694	0972	1250	1528						
	$\Phi'_{2,r}$	0015	0108	0293	0571	0941	1404						
n = 4	$\Phi'_{0,r}$	2500											
	$\Phi'_{1,r}$	0313	0938	1563	2188								
	$\Phi'_{2,r}$	0052	0365	0990	1927								

FIG. 4.—CHART OF FACTORS $\Phi'_{0,r}$, $\Phi'_{1,r}$, AND $\Phi'_{2,r}$ FOR USE IN COMPUTING FLEXURAL FUNCTIONS—MULTIPLY ALL VALUES BY 10^{-4}

By substitution into Eqs. 9,

$$F'_{1,0.5} = 0.500 - 0.167 = 0.333$$

and

$$F_{0,0.5} = \frac{0.011}{0.077} = 0.143.$$

These values may be substituted into Eq. 9c to yield $F_{1,0.5} = 0.205$.

For the second example, a beam having a compound soffit curve composed of a straight line and a tangent parabolic arc is chosen. Table 2 shows a sketch of the beam, and the complete computations for carry-over factors, stiffness factors, and fixed-end moment factors for uniform load. The computations are arranged in systematic order.

The flexural functions are evaluated by numerical integration, the beam being divided into six equal parts for the purpose. The ordinates t_t and t_p as well as the total ordinate t_r are computed at the center of each increment. Then, since the width of the beam is assumed uniform, the value of I_a/I_r is the value of $(t_a/t_r)^3$ for each increment. The thickness t_a is taken as 12.5 in., the first ordinate.

The values in lines 5, 8, and 11 of Table 2 are obtained by multiplying the values in line 4 by the factors in Fig. 4 for a value of $n = 6$ (six increments). It should be observed that the numbers in lines 6, 9, and 12 are each the sum of the numbers immediately above and to the right. The rest of the values in Table 2 are the numerical evaluation of the formulas given, and need no explanation.

To make the solution complete, it may be assumed that the slab shown in cross section is 1 ft wide and 35 ft long. Then, assuming E to be 1 kip per sq in.,

$$K_a = \frac{E\,I_a}{k_a\,L} = 24.4 \text{ kip-in. per ft}$$

$$K_b = \frac{E\,I_a}{k_b\,L} = 61.0 \text{ kip-in. per ft.}$$

To obtain the flexural functions for any given beam, divide the beam into n equal segments, n being a convenient arbitrary number. Then compute the value of I_a/I for each segment; I_a is a constant value of the moment of inertia—usually taken for the left-end segment. Multiply each value of I_a/I by the factors $\Phi'_{0,r}$, $\Phi'_{1,r}$, and $\Phi'_{2,r}$ for the corresponding segment and the same value of n. The values of these factors may be obtained from Fig. 4. The factors $\Phi_{0,r}$, $\Phi_{1,r}$, and $\Phi_{2,r}$ are obtained by summing these products from right to left (see lines 4 through 13 of Table 2, in which $n = 6$).

APPENDIX. NOTATION

The following symbols, adopted for use in the paper and for the guidance of discussers, comform essentially with American Standard Letter Symbols for Structural Analysis (ASA-Z10.8-1942), prepared by a Committee of the American Standards Association, with Society representation and approved by the Association in 1942:

C_{ab} = the carry-over factor at the small ($x = 0$)-end;

C_{ba} = the carry-over factor at the large ($x = 1$)-end;

E = the modulus of elasticity;

$\left.\begin{array}{l} F_{a,r} \\ F_{b,r} \end{array}\right\}$ = the fixed-end moment factors at the small and large ends, respectively, for a beam having both ends fixed and a concentrated load, P, at distance $r\,L$ from the small end, for use in the formula: (fixed-end moment) $= F\,P\,L$;

$\left.\begin{array}{l} F'_{a,r} \\ F'_{b,r} \end{array}\right\}$ = the fixed-end moment factors for a beam having one end hinged and the other end fixed and a concentrated load at distance $r\,L$ from the hinged end;

$\left.\begin{array}{l} F_{a,u} \\ F_{b,u} \\ F'_{a,u} \\ F'_{b,u} \end{array}\right\}$ = the fixed-end moment factors as the foregoing for a unit uniformly distributed load;

I_x = the moment of inertia at $x\,L$;

$\left.\begin{array}{l} k_a \\ k_b \end{array}\right\}$ = the dimensionless stiffness factors at the small and large ends, respectively, for use in the formula: Stiffness equals $K = \dfrac{E\,I_a}{k\,L}$;

L = the length of a member;

m = an integer signifying the degree of the flexural function;

N = a factor defined by Eq. 8;

n = the number of differential segments;

P = a concentrated load;

q = the segment designation;

$r\,L$ = the position coordinate for the concentrated load—constant during integration;

t_r = total depth of beam at a section;

$\left.\begin{array}{l} t_p \\ t_t \end{array}\right\}$ = depths of parts of a beam, as shown in Table 2;

$x\,L$ = the variable of integration;

Φ = a flexural function;

Φ' = a factor for computing Φ;

$\theta_{a,r}$ = the angle-of-rotation factor at the small end of the beam, corresponding to a unit load at a distance $r\,L$ from the small end, for use in the formula: $\theta = \theta_{a,r} \dfrac{P\,L^2}{E\,I_a}$, in which θ is the angle of rotation at the small end of the beam; and

$\theta_{a,u}$ = the angle-of-rotation factor at the small end corresponding to a unit uniform load.

(The subscript "a" refers to the small end of the beam shown in Fig. 1, at which end x equals zero, except that I_a is a "reference" value, taken at any stipulated section. The subscript "b" refers to the large end of the beam, at which x equals unity. The first subscript on the flexural functions Φ are the m-values that apply. The second subscript r on Φ, F, F', and θ corresponds to the position of the concentrated load and takes values 1 through n in sequence. A second subscript u on any symbol corresponds to a unit uniform load. The values of x and r are dimensionless.)